NOTATION: The author writes the letter f as "7" so 7 is not a seven but denotes the letter f

The solution of the first order linear partial differential equation is well covered in the literature. However, for an easily recognized special class of linear first order equations, we will present a simple solution which does not involve the solution of a system of ordinary differential equations. First, we need to state and prove the following:

Theorem:

IF $P = P(x,y)$ is an analytic function of (x,y) in some neighborhood D of the origin and if

$$P_{xy} = \frac{\partial^2 P}{\partial x \partial y} \equiv 0 \tag{1}$$

then

$$P(x,y) = P(x,0) + P(0,y) - P(0,0) \tag{2}$$

Proof:

Since $P_{xy} \equiv 0$, in D, it is clear that all other mixed partial derivatives are likewise identically zero in D. Hence Taylor's expansion for

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the analytic function $P(x,y)$ near the origin will reduce to the following:

$$P(x,y) = P(0,0) + P_x(0,0)x + P_y(0,0)y$$
$$+ \tfrac{1}{2}\left(P_{xx}(0,0)x^2 + P_{yy}(0,0)y^2\right)$$
$$+ \tfrac{1}{6}\left(P_{xxx}(0,0)x^3 + P_{yyy}(0,0)y^3\right) + \ldots \qquad (3)$$

$P(x,y)$, defined as above, will converge for all (x,y) IN D. So then if we let $y=0$ IN EQUA. (3) we obtain

$$P(x,0) = P(0,0) + P_x(0,0)x + \tfrac{1}{2}P_{xx}(0,0)x^2 +$$
$$+ \tfrac{1}{6}P_{xxx}(0,0)x^3 + \ldots \qquad (4)$$

Now setting $x=0$ IN EQUA. (3) we arrive at

$$P(0,y) = P(0,0) + P_y(0,0)y + \tfrac{1}{2}P_{yy}(0,0)y^2$$
$$+ \tfrac{1}{6}P_{yyy}(0,0)y^3 + \ldots \qquad (5)$$

ADDING EQUAS. (4) and (5) will give us

$$P(x,0) + P(0,y) = 2P(0,0) + P_x(0,0)x + P_y(0,0)y$$
$$+ \tfrac{1}{2}\left(P_{xx}(0,0)x^2 + P_{yy}(0,0)y^2\right)$$
$$+ \tfrac{1}{6}\left(P_{xxx}(0,0)x^3 + P_{yyy}(0,0)y^3\right) + \ldots \qquad (6)$$

1980

Subtracting $P(0,0)$ from both sides of EQUA. (6) results in the following:

$$P(x,0)+P(0,y)-P(0,0) = P(0,0)+P_x(0,0)x+P_y(0,0)y + \tfrac{1}{2}\left(P_{xx}(0,0)x^2+P_{yy}(0,0)y^2\right) + \tfrac{1}{6}\left(P_{xxx}(0,0)x^3+P_{yyy}(0,0)y^3\right)+\cdots \quad (7)$$

Now comparing the right sides of EQUAS. (7) and (3), we see that they are identical; Hence we may conclude that

$$P(x,y) = P(x,0)+P(0,y)-P(0,0)$$

as asserted.

We will now consider the linear first order Partial Differential Equation

$$\frac{\partial u}{\partial x} \pm \frac{\partial u}{\partial y} + P(x,y)u = 0$$

The above may be written more concisely as

$$u_x \pm u_y + Pu = 0 \quad (8)$$

Let $P(x,y)$ be analytic in some neighborhood D of the origin and let

$$P_{xy} \equiv 0 \quad (9)$$

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We will now propose the following as the general solution of EQUA. (8) subject to the condition of EQUA. (9):

$$u = EF(x \mp y) \qquad (10)$$

where

$$E = \exp\left(-\int\left(P(x,0) - \tfrac{1}{2}P(0,0)\right)dx \mp \int\left(P(0,y) - \tfrac{1}{2}P(0,0)\right)dy\right) \quad \ldots\ldots (11)$$

and where*

$$F = F(x \mp y) \qquad (12)$$

is an arbitrary differentiable function of $(x \mp y)$

Proof:
Partial differentiation of EQUAS. (11) and

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* Confusion regarding the $(\pm)$ and $(\mp)$ signs can be avoided by noting that when the sign of u_y in EQUA. (8) is $(+)$, then the sign of y in EQUA. (10) and the sign preceding the second integral of EQUA. (11) are both $(-)$ and vice versa.

 1980

(12) respectively with respect to x will result in the following:

$$E_x = -E\left(P(x,0) - \tfrac{1}{2}P(0,0)\right) \tag{13}$$

$$F_x = F'(x \mp y) = F' \tag{14}$$

Partial differentiation of EQUAS. (11) and (12) respectively with respect to y will result in

$$E_y = E\left(\mp\left(P(0,y) - \tfrac{1}{2}P(0,0)\right)\right) = \mp E\left(P(0,y) - \tfrac{1}{2}P(0,0)\right) \ldots\ldots (15)$$

$$F_y = \mp F'(x \mp y) = \mp F' \tag{16}$$

Now returning to EQUA. (10) and partially differentiating, in turn, with respect to x and y, we arrive at what follows:

$$u_x = E_x F + E F_x \tag{17}$$

$$u_y = E_y F + E F_y \tag{18}$$

Substituting the expressions for E_x and F_x (from EQUAS. (13) and (14)) into EQUA. (17) gives us

$$u_x = -E\left(P(x,0) - \tfrac{1}{2}P(0,0)\right)F + EF'$$

1980

from which we have

$$U_x = -E\left(\left(P(x,0) - \tfrac{1}{2}P(0,0)\right)F - F'\right) \quad \ldots\ldots\ldots (19)$$

Substituting the expressions for E_y and F_y (from EQUAS. (15) and (16)) into EQUA. (18) results in

$$U_y = \mp E\left(P(0,y) - \tfrac{1}{2}P(0,0)\right)F \mp EF'$$

which leads to

$$U_y = \mp E\left(\left(P(0,y) - \tfrac{1}{2}P(0,0)\right)F + F'\right) \quad \ldots\ldots\ldots (20)$$

Now substituting the expressions for U_x, U_y and U (from EQUAS. (19), (20) and (10) respectively) into EQUA. (8) will result in the following:

$$-E\left(\left(P(x,0) - \tfrac{1}{2}P(0,0)\right)F - F'\right) \pm \left(\mp E\left(\left(P(0,y) - \tfrac{1}{2}P(0,0)\right)F + F'\right)\right)$$

$$+PEF = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (21)$$

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Recalling the footnote on page 4, we see that

$$(\pm 1)(\mp 1) = -1$$

accordingly we may re-write EQUA.(21) as follows:

$$-E\Big(\big(P(x,0)-\tfrac{1}{2}P(0,0)\big)F - F'\Big) - E\Big(\big(P(0,y)-\tfrac{1}{2}P(0,0)\big)F + F'\Big)$$

$$+ PEF = 0$$

from which we obtain

$$-E\Big(\big(P(x,0)-\tfrac{1}{2}P(0,0)\big)F + \big(P(0,y)-\tfrac{1}{2}P(0,0)\big)F - PF\Big) = 0$$

dividing the above by the NON-VANISHING exponential E and then factoring out the COMMON factor F will give us

$$F\Big(\big(P(x,0)+P(0,y)-P(0,0)\big) - P\Big) = 0 \quad \ldots\ldots \quad (22)$$

Recalling that

$$P = P(x,0) + P(0,y) - P(0,0)$$

1980

When $P_{xy} \equiv 0$

we see that the left side of EQUA. (22) VANISHES thus establishing that

$$u = EF(x \mp y)$$

is the general solution of

$$u_x \pm u_y + Pu = 0$$

when $P_{xy} \equiv 0$ and where E is defined by EQUA. (11) and F is an arbitrary differentiable function of $(x \mp y)$

When we linearize
a non-linear transformation
we often realize
that it is an over-simplification

RHB

These concepts were informally distributed to selected members of the academic community in 1980

Your review and or critique will be appreciated

RHR

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